MULTIGRID WAVEFORM RELAXATION FOR DELAY PARTIAL DIFFERENTIAL EQUATIONS

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Abstract: The convergence of (multigrid) waveform relaxation methods is studied in a Fourier-Laplace framework for a specific class of functional-differential equations. This leads to quantitative convergence estimates, which are illustrated and validated by means of numerical experiments with two semi-discretised partial functional-differential model problems.

Keywords: Time delay, Differential equations, Partial differential equations, Numerical analysis, Iterative methods, Convergence of numerical methods

1. INTRODUCTION

Waveform relaxation is an iterative method originally developed for solving large systems of ordinary differential equations, see e.g. (Janssen, 1997; Miekkala and Nevanlinna, 1987; Miekkala and Nevanlinna, 1996). Recently, the method has also been applied to functional-differential equations, which arise for example in population dynamics and in the study of nonlinear materials with memory, (Bjørhus, 1994; Jackiewicz *et al.*, 1997; Zubik-Kowal and Vandewalle, 1999). In particular, the authors of (Zubik-Kowal and Vandewalle, 1999) introduced Picard waveform relaxation methods and derived error estimates which hold under certain Lipschitz conditions for the right-hand sides of the equations. Their results are illustrated by means of numerical experiments for semi-discrete partial functional-differential equations of parabolic type. Many examples of functional partial differential equations can be found in the monographs by Wu (Wu, 1996) and by Kolmanovskii and Myshkis (Kolmanovskii and Myshkis, 1999).

In this paper, quantitative convergence estimates of waveform relaxation methods for functional-differential equations are derived by analysing the methods in the classical Fourier-Laplace framework of Miekkala and Nevanlinna, (Miekkala and Nevanlinna, 1987). The analysis is illustrated for the numerical method of lines approximation of a two-dimensional diffusion equation extended with a term that has a constant delay in time. A similar Fourier-Laplace framework is used to investigate the acceleration of the basic waveform methods by multigrid techniques. Similar analyses have been done for multigrid waveform relaxation methods for non-functional parabolic partial differential equations, see e.g. (Janssen, 1997; Lubich and Ostermann, 1987; Vandewalle, 1993).

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The paper is organised as follows. In §2 the numerical method of lines approximation of the two-dimensional heat equation with a constant delay in time is considered. The Picard waveform relaxation method, introduced in (Zubik-Kowal and Vandewalle, 1999) is analysed in §3. Section 4 introduces and analyses non-Picard waveform relaxation methods. Multigrid waveform relaxation for the model problem is described in §5. Results of numerical experiments can be found in §6.

2. MODEL PROBLEM

Consider the two-dimensional parabolic functional-differential equation

$$\frac{\partial u(t,x,y)}{\partial t} = a\left(\frac{\partial^2 u(t,x,y)}{\partial x^2} + \frac{\partial^2 u(t,x,y)}{\partial y^2}\right) + bu(t-\tau_0,x,y), \quad (1)$$

for $(x,y) \in \Omega = [0,L]^2$, $t \in [0,T]$. The constants a and τ_0 are assumed to be positive. The initial and boundary conditions are chosen such that the exact solution becomes $u(t,x) = 0$. That is, $u(t,x) = 0$ for $(t,x,y) \in ([-\tau_0,0] \times \Omega) \cup ([0,T] \times \partial\Omega)$.

Applying to (1) the method of lines using a finite difference discretisation for the spatial dimensions leads to the following system of ordinary delay differential equations,

$$\dot{u}_{i,j}(t) = ah^{-2}\big(-4u_{i,j}(t) + u_{i-1,j}(t) + u_{i+1,j}(t) + u_{i,j-1}(t) + u_{i,j+1}(t)\big) + bu_{i,j}(t-\tau_0)$$

for $i,j = 1,\ldots,M$ and with $Mh = L$ and $u_{i,j}(t) \approx u(ih,jh,t)$. In matrix notation this gives

$$\dot{v}(t) + Av(t) + v(t-\tau_0) = 0 \quad (2)$$

for $t \in [0,T]$, with $v(t) = 0$ for $t \in [-\tau_0,0]$. Here, A is an $(M-1)^2 \times (M-1)^2$-matrix with the familiar block-tridiagonal structure with tridiagonal blocks. The vector $v(t)$ contains approximations $v_{(M-1)(i-1)+j}(t) \approx u(t,ih,jh)$ to the solution of the PDE.

For the examples the parameters are chosen such that (1) has a stable solution. The following result is used (Huang and Vandewalle, 2003, Corollary 3.3).

Theorem 1. The zero solution of (1) is asymptotically stable iff

- $a \geq 0$ and
- $-2\pi^2 a/L^2 < -b < \frac{\theta}{\tau_0 \sin\theta}$

where θ is the root of $\theta\cos\theta = -2\tau_0 a\pi^2/L^2$ such that $\pi/2 \leq \theta < \pi$.

Figure 1 shows the stability region in the (τ_0, L)-plane for $a = 1$ and $b = -1$. Points below the curve result in stable solutions. Figure 2 shows the stability region in the (τ_0, b)-plane for $a = 1$ and $L = 1$. Points between the curves result in stable solutions. In (Huang and Vandewalle, 2003) it is shown that the stability region of (2) is a subset of the stability region of (1). Because the difference is negligible for small mesh sizes h, the details are omitted here.

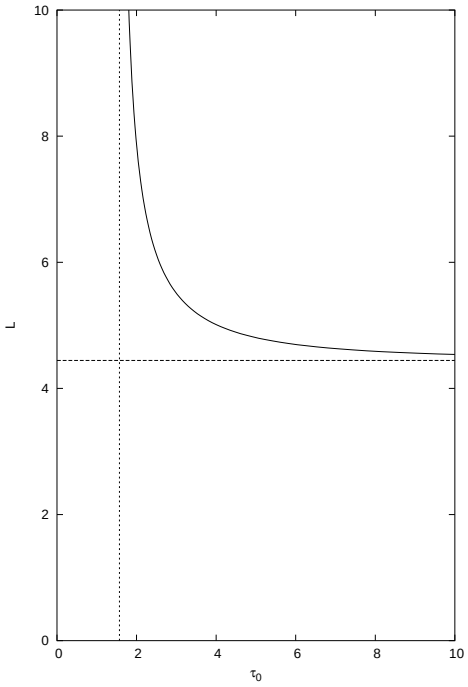

Fig. 1. Stability region in the (τ_0, L)-plane for $a = 1$ and $b = -1$.

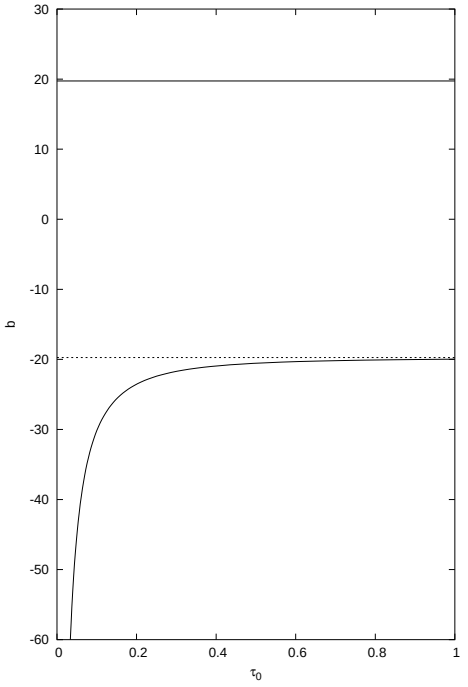

Fig. 2. Stability region in the (τ_0, b)-plane for $a = 1$ and $L = 1$.

To simplify the notation the parameters a and b are set to 1 and -1.

3. PICARD WAVEFORM RELAXATION

The *Picard waveform relaxation* methods for functional-differential equations that were introduced in (Zubik-Kowal and Vandewalle, 1999) can be applied to (2). This gives, for $t \in [0, T]$,

$$\dot{v}^{(\nu)}(t) + M_A v^{(\nu)}(t) = N_A v^{(\nu-1)}(t) - v^{(\nu-1)}(t - \tau_0) \quad (3)$$

where $v^{(\nu)}(t) = 0$ for $t \in [-\tau_0, 0]$ and $A = M_A - N_A$. The term 'Picard' is used to stress that the functional argument is taken from a previous iterate in all these schemes. The *direct/Picard* method is defined by $M_A = A$, $N_A = 0$, while, in terms of the classical splitting $A = D_A - L_A - U_A$ in diagonal, lower triangular and upper triangular parts, the *Jacobi/Picard* and *Gauss-Seidel/Picard* waveform relaxation schemes correspond to $M_A = D_A$, $N_A = L_A + U_A$ and $M_A = D_A - L_A$, $N_A = U_A$, respectively.

In (Zubik-Kowal and Vandewalle, 1999), the convergence of the Picard waveform relaxation methods was analysed for general ordinary functional-differential equations by deriving error estimates which depend on some Lipschitz conditions of the right-hand side. While such an analysis gives rise to qualitative results, the current model problem also allows a quantitative analysis of the waveform methods in the Fourier-Laplace framework developed by Miekkala and Nevanlinna, (Miekkala and Nevanlinna, 1987).

To start the analysis of the Picard waveform relaxation methods, (2) is rewritten as

$$\dot{v}(t) + M_A v(t) = N_A v(t) - v(t - \tau_0) \ , \quad t \in [0, T] \ .$$

Subtracting this equation from (3) gives the error equivalent of the latter,

$$\dot{e}^{(\nu)}(t) + M_A e^{(\nu)}(t) = N_A e^{(\nu-1)}(t) - e^{(\nu-1)}(t - \tau_0), \quad t \in [0, T], \quad (4)$$

where $e^{(\nu)}(t) = v^{(\nu)}(t) - v(t)$. If the Laplace transform of $e^{(\nu)}(t)$ is denoted by $\widetilde{e}^{(\nu)}(z)$, Laplace transforming (4) results in

$$\widetilde{e}^{(\nu)}(z) = \mathbf{K}(z)\widetilde{e}^{(\nu-1)}(z), \quad (5)$$

with the *Picard waveform relaxation symbol*

$$\mathbf{K}(z) = (zI + M_A)^{-1}(-e^{-\tau_0 z} I + N_A) \ . \quad (6)$$

By an inverse Laplace-transform argument, the relation,

$$e^{(\nu)}(t) = \mathcal{K} e^{(\nu-1)}(t) = \int_0^t k(t - s) e^{(\nu-1)}(s) ds \ , \quad (7)$$

is derived, i.e., the *Picard waveform relaxation operator* $\mathcal{K}$ is a linear Volterra convolution operator with kernel $k(t)$. More precisely, since $\lim_{z\to\infty} \mathbf{K}(z) = 0$, $k(t) \in L_1(0, \infty)$ if $\mathbf{K}(z)$ is bounded and analytic in an open domain containing the closed right-half complex plane, i.e., if all eigenvalues of M_A have positive real parts, see e.g. (Jordan *et al.*, 1982, Prop. 2.3).

The spectral radius $\rho(\mathcal{K})$ of this convolution operator $\mathcal{K}$ determines the asymptotic convergence of the corresponding Picard waveform relaxation method, and can be investigated in the spaces $L_p(0, \infty)$ of Lebesgue-measurable functions which are p-th power integrable, see e.g. (Janssen, 1997, Lemma 2.2.3).

Theorem 2. Consider equation (1), discretised in space using the numerical method of lines, and assume all eigenvalues of M_A have positive real parts. Then, the spectral radius of the Picard waveform relaxation operator $\mathcal{K}$ considered as an operator in $L_p(0, \infty)$ with $1 \le p \le \infty$ is given by

$$\rho(\mathcal{K}) = \sup_{\mathrm{Re}(z) \ge 0} \rho(\mathbf{K}(z)) \quad (8)$$

$$= \sup_{\xi \in \mathbb{R}} \rho(\mathbf{K}(i\xi)) \ , \quad (9)$$

with $\mathbf{K}(z)$ as in (6).

4. WAVEFORM RELAXATION METHODS

The standard (that is, non-Picard) waveform relaxation methods for (2) are given by

$$\dot{v}^{(\nu)}(t) + M_A v^{(\nu)}(t) + v^{(\nu)}(t - \tau_0) = N_A v^{(\nu-1)}(t), \quad t \in [0, T]. \quad (10)$$

The *Jacobi* and *Gauss-Seidel* variants of which are defined in terms of their respective splittings given above.

The non-Picard waveform relaxation methods (10) can be analysed in a similar way as the Picard methods. Rewriting (2) as

$$\dot{v}(t) + M_A v(t) + v(t - \tau_0) = N_A v(t), \quad t \in [0, T],$$

subtracting this from (10) and applying Laplace transforms, results in (5) with the *waveform relaxation symbol*

$$\mathbf{K}(z) = ((z + e^{-\tau_0 z})I + M_A)^{-1} N_A. \quad (11)$$

An inverse Laplace-transform argument then immediately implies that the *waveform relaxation operator* $\mathcal{K}$ is a linear Volterra convolution operator with kernel $k(t)$ (see (7)).

In order to be able to determine the spectral radius $\rho(\mathcal{K})$ in the spaces $L_p(0, \infty)$, one has to prove that $k(t) \in L_1(0, \infty)$. From (Jordan *et al.*, 1982, Prop. 2.3), it follows that it suffices that

$\lim_{z\to\infty}\mathbf{K}(z)=0$ and that $\mathbf{K}(z)$ is bounded and analytic in an open domain containing the closed right-half complex plane. The first condition is easily checked. The second one demands

$$\det((z+e^{-\tau_0 z})I+M_A)\neq 0,\quad \mathrm{Re}(z)\geq 0$$

or equivalently

$$\det((-z-e^{-\tau_0 z})I-M_A)\neq 0,\quad \mathrm{Re}(z)\geq 0. \quad (12)$$

Since

$$\max\{\mathrm{Re}(-z-e^{-\tau_0 z})\mid \mathrm{Re}(z)\geq 0 \text{ and } \mathrm{Im}(-z-e^{-\tau_0 z})=0\}<0. \quad (13)$$

for any $\tau_0\geq 0$, the former condition is fulfilled if all eigenvalues of M_A are real and positive.

Theorem 3. Consider equation (1), discretised in space using the numerical method of lines, and assume all eigenvalues of M_A are real and positive. Then, the spectral radius of the waveform relaxation operator $\mathcal{K}$ considered as an operator in $L_p(0,\infty)$ with $1\leq p\leq\infty$, is given by

$$\begin{aligned}\rho(\mathcal{K}) &= \sup_{\mathrm{Re}(z)\geq 0}\rho(\mathbf{K}(z)),\\ &= \sup_{\xi\in\mathbb{R}}\rho(\mathbf{K}(i\xi)),\end{aligned} \quad (14)$$

with $\mathbf{K}(z)$ as in (11).

5. MULTIGRID WAVEFORM RELAXATION

The convergence rate of the Jacobi and Gauss-Seidel waveform relaxation methods, described in the previous sections, depends on the spatial mesh size. If they converge, the convergence will be very slow except on very coarse grids. This is to be expected since the same happens for the corresponding problems without delay. However, just as in the case without delay, efficient multigrid methods can be developed based on the simple waveform relaxation schemes. Below, a two-grid cycle for model problem (2) is stated. A Picard waveform relaxation method is used as a smoother. Two nested grids are used. The quantities on the fine grid are denoted by the subscript h, those on the coarse grid by the subscript H.

- Pre-smoothing. Set $x_h^{(0)}=v_h^{(\nu-1)}$, and perform ν_1 waveform relaxation/Picard steps: for $\nu=1,2,\ldots,\nu_1$, solve

$$\dot{x}_h^{(\nu)}(t)+M_{A_h}x_h^{(\nu)}(t)= N_{A_h}x_h^{(\nu-1)}(t)-x_h^{(\nu-1)}(t-\tau_0), \quad (15)$$

  with $x_h^{(\nu)}(t)=0$, $t\in[-\tau_0,0]$.
- Coarse-grid correction. Compute the defect

$$d_h(t)=\dot{x}_h^{(\nu_1)}(t)+A_h x_h^{(\nu_1)}(t)+x_h^{(\nu_1)}(t-\tau_0).$$

  Solve the coarse-grid equivalent of the defect equation,

$$\dot{v}_H(t)+A_H v_H(t)-v_H(t-\tau_0)=r d_h(t), \quad (16)$$

  with $v_H(t)=0$, $t\in[-\tau_0,0]$ and r the *restriction operator* transferring fine-grid quantities to coarse-grid ones. Then, interpolate the correction $v_H(t)$ to the fine grid, and correct the current approximation,

$$\bar{x}_h(t)=x_h^{(\nu_1)}(t)-p v_H(t),$$

  with p the *prolongation operator* which projects coarse-grid quantities onto fine-grid ones.
- Post-smoothing. Perform ν_2 iterations of type (15), starting with $x_h^{(0)}(t)=\bar{x}_h(t)$, and set $v_h^{(\nu)}(t)=x_h^{(\nu_2)}(t)$.

Since (16) is formally equal to (2) (except for the zero right-hand side in the latter problem), this two-grid cycle can be applied in a recursive way to obtain a multigrid cycle on more than two nested grids.

The theoretical analysis of the two-grid waveform relaxation method can be performed in analogy with the analysis of the (Picard) waveform relaxation methods. Rewriting all the steps of the two-grid scheme in error notation and applying Laplace-transform techniques results in

$$\tilde{e}_h^{(\nu)}(z)=\mathbf{M}(z)\tilde{e}_h^{(\nu-1)}(z), \quad (17)$$

with the *two-grid symbol*

$$\mathbf{M}(z)=\mathbf{K}^{\nu_2}(z)\mathbf{C}(z)\mathbf{K}^{\nu_1}(z), \quad (18)$$

$$\mathbf{C}(z)=I-p\mathbf{L_H}(z)^{-1}r\mathbf{L_h}(z), \quad (19)$$

$$\mathbf{L_H}(z)=(z+e^{-\tau_0 z})I+A_H, \quad (20)$$

$$\mathbf{L_h}(z)=(z+e^{-\tau_0 z})I+A_h, \quad (21)$$

$$\mathbf{K}(z)=(zI+M_{A_h})^{-1}(-e^{-\tau_0 z}I+N_{A_h}). \quad (22)$$

An inverse Laplace-transform argument shows that the *two-grid waveform relaxation operator* $\mathcal{M}$ is of linear Volterra convolution type, i.e.,

$$e_h^{(\nu)}=\mathcal{M}e_h^{(\nu-1)}=\int_0^t m(t-s)e_h^{(\nu-1)}(s)ds. \quad (23)$$

In order to be able to determine the spectral radius $\rho(\mathcal{M})$ in the spaces $L_p(0,\infty)$, it has to be shown that $m(t)\in L_1(0,\infty)$. From (Jordan *et al.*, 1982, Prop. 2.3), it follows that to this end, it suffices that $\lim_{z\to\infty}\mathbf{M}(z)=0$ and that $\mathbf{M}(z)$ is bounded and analytic in an open domain containing the closed right-half of the complex plane. The first condition follows immediately from the fact that $\lim_{z\to\infty}\mathbf{K}(z)=0$. On the other hand, $\mathbf{M}(z)$ is bounded and analytic in an open domain containing the closed right-half of the complex plane if $\mathbf{K}(z)$ and $\mathbf{C}(z)$ are both bounded and analytic in such a region. This means

Table 1. Values of $\rho(\mathcal{M})$ for $a = 1$, $b = -1$, $M = 32$

τ_0	L	Picard	non-Picard
1	1	0.1625	0.1624
1	2	0.1626	0.1620
1	5	0.1651	0.1630
2	1	0.1623	0.1625
2	2	0.1627	0.1625
2	5	0.1637	0.1650

that all eigenvalues of M_{A_h} must have positive real parts and that

$$\det((z + e^{-\tau_0 z})I + A_H) \neq 0, \quad \mathrm{Re}(z) \geq 0.$$

The latter is trivially satisfied as the eigenvalues of A_H are positive and (13) holds. The following theorem therefore holds.

Theorem 4. Consider equation (1), discretised in space using the numerical method of lines, and assume all eigenvalues of M_{A_h} have positive real parts. The spectral radius of the two-grid waveform relaxation operator $\mathcal{M}$, considered as an operator in $L_p(0, \infty)$ with $1 \leq p \leq \infty$, is given by

$$\rho(\mathcal{M}) = \sup_{\mathrm{Re}(z) \geq 0} \rho(\mathbf{M}(z)) \tag{24}$$

$$= \sup_{\xi \in \mathbb{R}} \rho(\mathbf{M}(i\xi)), \tag{25}$$

with $\mathbf{M}(z)$ as in (22).

Figure 3 shows the spectral radius of the two-grid symbol $\rho(\mathbf{M}(z))$ over the imaginary axis for (2), with $L = 10$, $a = 1$, $b = -1$, $\tau_0 = 1$, $M = 32$ and using Gauss-Seidel/Picard waveform relaxation as smoother. The scale of the horizontal axis is not specified since it can easily be shown that, except for the scale, the result is exactly the same for all parameters for which $a\tau_0 L^{-2} = 10^{-2}$ and $b\tau_0 = -1$. The delay manifests itself as a wiggle on top of the curve for the equation without delay ($b = 0$). Its amplitude and frequency depend on the choice of parameters.

Instead of a Picard waveform relaxation method, a standard waveform relaxation method can be used as smoother, i.e., (15) can be replaced by

$$\dot{x}_h^{(\nu)}(t) + M_{A_h} x_h^{(\nu)}(t) + x_h^{(\nu)}(t - \tau_0) = N_{A_h} x_h^{(\nu-1)}(t).$$

If all eigenvalues of M_{A_h} are real and positive, similar results as in Theorem 4 can be proven for the resulting multigrid waveform operator $\mathcal{M}$.

Table 1 shows the spectral radii for $a = 1$, $b = -1$, $M = 32$ and several values of L and τ_0.

6. NUMERICAL RESULTS

Some numerical results are presented for the multigrid waveform relaxation method applied to (1),

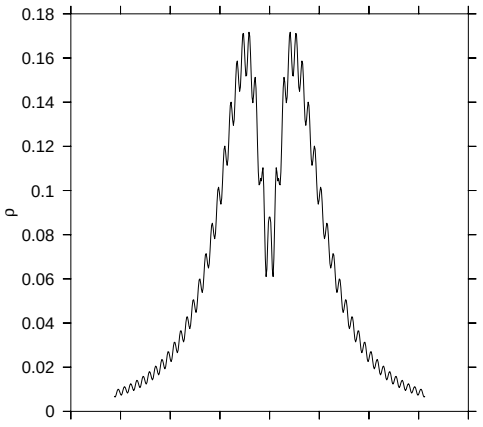

Fig. 3. $\rho(\mathbf{M}(i\xi))$ for $L = 10$, $a = 1$, $b = -1$, $\tau_0 = 1$.

Table 2. Averaged convergence factors for $a = 1$, $b = -1$, $M = 32$, $T = 50$

τ_0	L	Picard	non-Picard
1	1	0.1374	0.1161
1	2	0.1951	0.1477
1	5	0.2037	0.1797
2	1	0.1372	0.1168
2	2	0.2021	0.1124
2	5	0.1157	0.1089

discretised using the method of lines. Instead of solving (16) exactly on the coarse grid, the two-grid method is implemented in a recursive manner until a mesh with 1 internal grid point is obtained. The method uses a standard V-cycle with 1 pre- and 1 post-smoothing step of Gauss-Seidel/Picard or Gauss-Seidel type, linear interpolation and full-weighting restriction. Obviously, the continuous-time method has to be discretised in time in an actual implementation. The effect of this time discretisation on the convergence properties is not considered here. The backward differentiation formula of order two with time-step 0.1 is used for all the experiments. Initially all unknowns are set to one.

Table 2 reports averaged convergence factors for the parameters $a = 1$, $b = -1$, $M = 32$, on a time interval with length $T = 50$. The averages are obtained by taking the geometric average of quotient of the L_2-norm of consecutive errors for the last 20 iterations in a total of 40. These factors correspond roughly to the infinite-interval results of Table 1. For large values of L the convergence becomes slightly more erratic, but the methods are still efficient.

To show that it is possible to solve more general equations using the methods discussed here, a multigrid method was implemented to solve a general diffusion equation with varying coefficients and a term with a constant delay.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(b\frac{\partial u}{\partial y}\right) + cu + du(t-\tau_0) + f,$$
$$a(x,y,t) = \exp(10(x-y)\sin(t)),$$
$$b(x,y,t) = \exp(-10(x-y)\cos(\pi t)),$$
$$c(x,y,t) = 2 - \exp(-t),$$
$$d(x,y,t) = 1 + \exp(t).$$

The function f is chosen such that the exact solution is $u(x,y,t) = x + y + t$. This is an anisotropic problem since the diffusion coefficients in a certain point can be very different depending on the direction of diffusion. Standard multigrid methods do not handle this type of problem well and yield very slow convergence. The "multigrid as smoother" (MGS) method, which uses the same simple smoothers as before, together with an extended hierarchy of coarse grids, was used here. For more information on multigrid for anisotropic problems see (Trottenberg *et al.*, 2001; Van lent and Vandewalle, 2002; Oosterlee and Wesseling, 1995; Washio and Oosterlee, 1998; Oosterlee, 1995). The multigrid algorithm uses full weighting restriction, bilinear and linear interpolation, 1 pre- and 1 postsmoothing step. The smoothing consists of a multigrid semicoarsening step in the x-direction followed by one in the y-direction both using only 1 presmoothing red-black Gauss–Seidel step. A hierarchy of 5×5 grids is used. A total of 10 iterations was performed. Average convergence factors, taken over the last 5 iterations, are 0.0557 for the Picard as well as the non-Picard smoother. The partial differential equation is solved to within the spatial discretisation error in 4 iterations (there is no discretisation error in time).

7. CONCLUSION

In this paper, a theoretical Fourier-Laplace framework is set up to derive quantitative convergence estimates of waveform relaxation methods for semi-discretised partial functional-differential equations.

Roughly speaking, a great resemblance with the convergence behaviour of the related methods for partial differential equations without functional argument can be observed, see e.g. (Janssen, 1997). That is, the multigrid waveform relaxation methods exhibit mesh-size independent convergence behaviour.

Although a very simple model problem is used to explain and illustrate the methods and their analysis, the provided information should allow one to derive similar results for more complicated problems. Other issues which one might be interested in are, e.g. the influence of the particular time discretisation method used on the algorithm's convergence speed, and the treatment of non-linearities and more general, variable or state-dependent delays.

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