

Convolution SOR waveform relaxation on spatial finite element meshes

This paper investigates the convergence properties of the convolution SOR waveform relaxation method, applied to a system of ordinary differential equations, obtained by spatial finite element discretisation of a linear parabolic initial boundary value problem. We consider both the continuous-time and discrete-time cases and provide a model problem analysis for the one-dimensional heat equation.

1. Introduction

We consider the numerical solution of a linear parabolic initial boundary value problem, spatially discretised by a conforming Galerkin finite element method. This leads to a linear system of d ordinary differential equations (ODEs)

$$B\dot{u}(t) + Au(t) = f(t), \quad u(0) = u_0, \quad (1)$$

with $B = (b_{ij})_{i,j=1}^d$ the symmetric positive definite mass matrix and $A = (a_{ij})_{i,j=1}^d$ the stiffness matrix, [7]. For such systems of ODEs, the standard waveform relaxation method and its multigrid acceleration are investigated in [2, 3]. In this paper, we study the convolution SOR (CSOR) waveform relaxation method, introduced in [6] by Reichelt et al., for general ODE-systems of the form (1). This paper is a summary of [4], where a more detailed study of the method, including proofs and a more extensive reference list, can be found. We begin in §2 by describing the continuous-time and discrete-time CSOR waveform relaxation algorithms. The convergence analysis of these methods is outlined in §3. The paper ends in §4, where theoretical and numerical results are given for the one-dimensional heat equation.

2. CSOR waveform relaxation algorithms

The CONTINUOUS-TIME CSOR waveform relaxation algorithm for system (1) computes the new waveform approximation $u_i^{(\nu)}(t)$, $1 \leq i \leq d$, from the previous approximation in two steps. The first step consists of a Gauss-Seidel like computation of $\hat{u}_i^{(\nu)}(t)$ along a continuous time-interval,

$$\left(b_{ii} \frac{d}{dt} + a_{ii}\right) \hat{u}_i^{(\nu)}(t) = - \sum_{j=1}^{i-1} \left(b_{ij} \frac{d}{dt} + a_{ij}\right) u_j^{(\nu)}(t) - \sum_{j=i+1}^d \left(b_{ij} \frac{d}{dt} + a_{ij}\right) u_j^{(\nu-1)}(t) + f_i(t). \quad (2)$$

In the second step the old approximation $u_i^{(\nu-1)}(t)$ is updated. Whereas a standard SOR method would extrapolate the correction by multiplying with an overrelaxation parameter ω , [4], here we convolute the correction with a time-dependent kernel $\Omega(t)$,

$$u_i^{(\nu)}(t) = u_i^{(\nu-1)}(t) + \int_0^t \Omega(t-\tau) \cdot \left(\hat{u}_i^{(\nu)}(\tau) - u_i^{(\nu-1)}(\tau)\right) d\tau. \quad (3)$$

The DISCRETE-TIME CSOR waveform relaxation algorithm is obtained, for example, by applying a linear multistep method, [1], to (2). Hence, the first step becomes

$$\begin{aligned} \sum_{l=0}^k \left(\frac{1}{\tau} \alpha_l b_{ii} + \beta_l a_{ii}\right) \hat{u}_i^{(\nu)}[n+l] &= - \sum_{j=1}^{i-1} \sum_{l=0}^k \left(\frac{1}{\tau} \alpha_l b_{ij} + \beta_l a_{ij}\right) u_j^{(\nu)}[n+l] \\ &\quad - \sum_{j=i+1}^d \sum_{l=0}^k \left(\frac{1}{\tau} \alpha_l b_{ij} + \beta_l a_{ij}\right) u_j^{(\nu-1)}[n+l] + \sum_{l=0}^k \beta_l f_i[n+l], \end{aligned} \quad (4)$$

where α_l and β_l are the coefficients of the multistep method, τ is the (constant) step-size, and $u_i^{(\nu)}[n]$ denotes the discrete approximation of $u_i^{(\nu)}(t)$ at $t = n\tau$. In the second step, the continuous-time convolution is replaced by its

discrete equivalent,

$$u_i^{(\nu)}[n] = u_i^{(\nu-1)}[n] + \sum_{l=0}^n \Omega[n-l] \cdot \left(\hat{u}_i^{(\nu)}[l] - u_i^{(\nu-1)}[l] \right). \quad (5)$$

3. Convergence analysis

By rewriting the CONTINUOUS-TIME iterative scheme (2)–(3) in explicit form, we can derive that the CSOR iteration operator \mathcal{K}^{CSOR} consists of a matrix multiplication and a linear Volterra convolution part,

$$u^{(\nu)}(t) = \mathcal{K}^{CSOR} u^{(\nu-1)}(t) + \varphi(t) = K^{CSOR} u^{(\nu-1)}(t) + \int_0^t k_c^{CSOR}(t-\tau) u^{(\nu-1)}(\tau) d\tau + \varphi(t). \quad (6)$$

Laplace-transformation of (6) leads to $\tilde{u}^{(\nu)}(z) = \mathbf{K}^{CSOR}(z) \tilde{u}^{(\nu-1)}(z) + \tilde{\varphi}(z)$, where $\tilde{u}^{(\nu)}(z)$ denotes the Laplace-transform of the function $u^{(\nu)}(t)$. The continuous-time CSOR symbol equals

$$\mathbf{K}^{CSOR}(z) = \left(z \left(\frac{1}{\tilde{\Omega}(z)} D_B - L_B \right) + \left(\frac{1}{\tilde{\Omega}(z)} D_A - L_A \right) \right)^{-1} \cdot \left(z \left(\frac{1 - \tilde{\Omega}(z)}{\tilde{\Omega}(z)} D_B + U_B \right) + \left(\frac{1 - \tilde{\Omega}(z)}{\tilde{\Omega}(z)} D_A + U_A \right) \right),$$

with $B = D_B - L_B - U_B$ and $A = D_A - L_A - U_A$ the standard splittings of B and A in diagonal, lower and upper triangular parts. In terms of this symbol, we can prove the following convergence theorem on infinite time-intervals.

Theorem 1. *Let \mathcal{K}^{CSOR} be an operator in $L_p(0, \infty)$, $1 \leq p \leq \infty$. Then, \mathcal{K}^{CSOR} is a bounded operator and*

$$\rho(\mathcal{K}^{CSOR}) = \sup_{\operatorname{Re}(z) \geq 0} \rho(\mathbf{K}^{CSOR}(z)) = \sup_{\xi \in \mathbb{R}} \rho(\mathbf{K}^{CSOR}(i\xi)). \quad (7)$$

The following expression for the Laplace-transform of the optimal convolution kernel $\Omega_{opt}(t)$ can be derived, [4].

Theorem 2. *Assume B and A are such that $zB + A$ is consistently ordered, $\det(zD_B + D_A) \neq 0$, and the spectrum $\sigma(\mathbf{K}^{JAC}(z)) = \sigma((zD_B + D_A)^{-1}(z(L_B + U_B) + (L_A + U_A)))$ lies on a line segment $[-\mu_1(z), \mu_1(z)]$ with $\mu_1(z) \in \mathbb{C}$ and $|\mu_1(z)| < 1$. The spectral radius of $\mathbf{K}^{CSOR}(z)$ is then minimised by the unique optimum $\tilde{\Omega}_{opt}(z)$, and is given by*

$$\rho(\mathbf{K}^{CSOR, \tilde{\Omega}_{opt}(z)}(z)) = |\tilde{\Omega}_{opt}(z) - 1| < 1, \quad \text{with } \tilde{\Omega}_{opt}(z) = \frac{2}{1 + \sqrt{1 - \mu_1^2(z)}}, \quad (8)$$

where $\sqrt{\cdot}$ denotes the root with the positive real part.

A similar analysis can be done for the DISCRETE-TIME CSOR waveform relaxation method. The operator \mathcal{K}_τ^{CSOR} turns out to be of discrete convolution type, as we can rewrite (4)–(5) into the following form

$$u^{(\nu)}[n] = \left(\mathcal{K}_\tau^{CSOR} u_\tau^{(\nu-1)} \right)[n] + \varphi[n] = \sum_{i=0}^n k[n-l] u^{(\nu-1)}[l] + \varphi[n]. \quad (9)$$

The discrete-time CSOR symbol $\mathbf{K}_\tau^{CSOR}(z)$ is obtained by discrete Laplace- or Z -transformation of (9). More precisely, we have $\tilde{u}_\tau^{(\nu)}(z) = \mathbf{K}_\tau^{CSOR}(z) \tilde{u}_\tau^{(\nu-1)}(z) + \tilde{\varphi}_\tau(z)$ where $\tilde{u}_\tau^{(\nu)}(z)$ denotes the Z -transform of the sequence $u_\tau^{(\nu)} = \{u^{(\nu)}[0], u^{(\nu)}[1], u^{(\nu)}[2], \dots\}$ and

$$\mathbf{K}_\tau^{CSOR}(z) = \left(\frac{1}{\tau} \frac{a}{b}(z) \left(\frac{1}{\tilde{\Omega}_\tau(z)} D_B - L_B \right) + \left(\frac{1}{\tilde{\Omega}_\tau(z)} D_A - L_A \right) \right)^{-1} \cdot \left(\frac{1}{\tau} \frac{a}{b}(z) \left(\frac{1 - \tilde{\Omega}_\tau(z)}{\tilde{\Omega}_\tau(z)} D_B + U_B \right) + \left(\frac{1 - \tilde{\Omega}_\tau(z)}{\tilde{\Omega}_\tau(z)} D_A + U_A \right) \right),$$

with $a(z) = \sum_{j=0}^k \alpha_j z^j$ and $b(z) = \sum_{j=0}^k \beta_j z^j$ the characteristic polynomials of the multistep method. We then obtain the following discrete-time equivalents of Theorems 1 and 2.

Theorem 3. Let \mathcal{K}_τ^{CSOR} be an operator in $l_p(\infty)$, $1 \leq p \leq \infty$. Then \mathcal{K}_τ^{CSOR} is a bounded operator and

$$\rho(\mathcal{K}_\tau^{CSOR}) = \max_{|z| \geq 1} \rho(\mathbf{K}_\tau^{\text{CSOR}}(z)) = \max_{|z|=1} \rho(\mathbf{K}_\tau^{\text{CSOR}}(z)) . \quad (10)$$

Theorem 4. Assume B and A are such that $\frac{1}{\tau} \frac{a}{b}(z)B + A$ is consistently ordered, $\det(\frac{1}{\tau} \frac{a}{b}(z)D_B + D_A) \neq 0$, and the spectrum $\sigma(\mathbf{K}_\tau^{\text{JAC}}(z)) = \sigma(\mathbf{K}^{\text{JAC}}(\frac{1}{\tau} \frac{a}{b}(z)))$ lies on a line segment $[-(\mu_1)_\tau(z), (\mu_1)_\tau(z)]$ with $(\mu_1)_\tau(z) \in \mathbb{C}$ and $|(\mu_1)_\tau(z)| < 1$. The spectral radius of $\mathbf{K}_\tau^{\text{CSOR}}(z)$ is then minimised by the unique optimum $(\tilde{\Omega}_{opt})_\tau(z)$, and is given by

$$\rho(\mathbf{K}_\tau^{\text{CSOR}, (\tilde{\Omega}_{opt})_\tau(z)}(z)) = |(\tilde{\Omega}_{opt})_\tau(z) - 1| < 1 \quad , \quad \text{with} \quad (\tilde{\Omega}_{opt})_\tau(z) = \frac{2}{1 + \sqrt{1 - (\mu_1)_\tau^2(z)}} , \quad (11)$$

where $\sqrt{\cdot}$ denotes the root with the positive real part.

By comparison of (8) and (11), we observe that $(\tilde{\Omega}_{opt})_\tau(z) = \tilde{\Omega}_{opt}(\frac{1}{\tau} \frac{a}{b}(z))$. Consequently, in the optimal case, (10) can be rewritten as

$$\rho(\mathcal{K}_\tau^{CSOR, (\Omega_{opt})_\tau}) = \sup \left\{ \rho(\mathbf{K}^{\text{CSOR}, \tilde{\Omega}_{opt}(z)}(z)) \mid \tau z \in \mathbb{C} \setminus \text{int}S \right\} = \sup_{z \in \frac{1}{\tau} \partial S} \rho(\mathbf{K}^{\text{CSOR}, \tilde{\Omega}_{opt}(z)}(z)) , \quad (12)$$

where S denotes the stability region of the multistep method.

4. Model problem analysis

Consider the one-dimensional heat equation

$$\frac{\partial \mathbf{u}}{\partial t} - \Delta_1 \mathbf{u} = 0 , \quad x \in [0, 1] , \quad t \in [0, 1] , \quad (13)$$

with homogeneous Dirichlet boundary conditions and a given initial condition. Let the equation be discretised using linear finite element basis functions on a grid with mesh-size h , i.e., $\Omega_h = \{x_i = ih \mid 0 \leq i \leq 1/h\}$. This leads to a system of ODEs (1) with (in stencil notation) $B = [\frac{h}{8} \quad \frac{4h}{8} \quad \frac{h}{8}]$ and $A = [\frac{-1}{h} \quad \frac{2}{h} \quad \frac{-1}{h}]$.

Theorem 5. Consider $\mathcal{K}^{CSOR, \Omega_{opt}(t)}$ as an operator in $L_p(0, \infty)$, $1 \leq p \leq \infty$, for solving (13), discretised using linear finite element basis functions. Then, for small h , we have that

$$\rho(\mathcal{K}^{CSOR, \Omega_{opt}(t)}) \approx 1 - 2\pi h . \quad (14)$$

The proof of Theorem 5 is based on the observation that the maximum of $\rho(\mathbf{K}^{\text{CSOR}, \tilde{\Omega}_{opt}(z)}(z))$ along the imaginary axis is attained at the origin. Hence, $\rho(\mathcal{K}^{CSOR, \Omega_{opt}(t)})$ equals the spectral radius of the algebraic SOR method for matrix A with optimal overrelaxation parameter $\tilde{\Omega}_{opt}(0)$, which is well-known to be $1 - 2\pi h$ for small h , [8].

Table 1 presents some averaged convergence factors obtained with an implementation of the discrete-time convolution SOR waveform relaxation method with optimal convolution kernel, for model problem (13). We used the Crank-Nicolson (CN) method and the backward differentiation (BDF) formulae of order 1, 3 and 5, with time-step $\tau = 1/100$. It is well-known for waveform relaxation methods that the numerical results, although obtained on finite time-intervals, match the infinite time-interval theoretical analysis, see e.g. [5] for a theoretical explanation based on the pseudospectra of the relevant operators. Indeed, the observed convergence factors in Table 1 are in close correspondence with the theoretical result (14). Also, the results of Table 1 show that, for a fixed value of h , the observed convergence factors are more or less independent of the chosen time-discretisation method. An explanation of this behaviour can be found in Figure 1, where we visualise the application of formula (12) for model problem (13) by means of a so-called spectral picture: we display contour lines of $\rho(\mathbf{K}^{\text{CSOR}, \tilde{\Omega}_{opt}(z)}(z))$ (for values

h	1/8	1/16	1/32	1/64
CN	0.388	0.637	0.802	0.899
BDF(1)	0.389	0.637	0.802	0.896
BDF(3)	0.378	0.630	0.797	0.894
BDF(5)	0.377	0.629	0.797	0.894
$1 - 2\pi h$	0.215	0.607	0.804	0.902

Table 1: Observed convergence factors for (13) – optimal CSOR waveform relaxation – $\tau = 1/100$.

0.6, 0.7, 0.8 and 0.9), together with (parts of) the scaled stability region boundaries $\frac{1}{\tau}\partial S$ of the CN method and the BDF methods of order 1, 3 and 5.

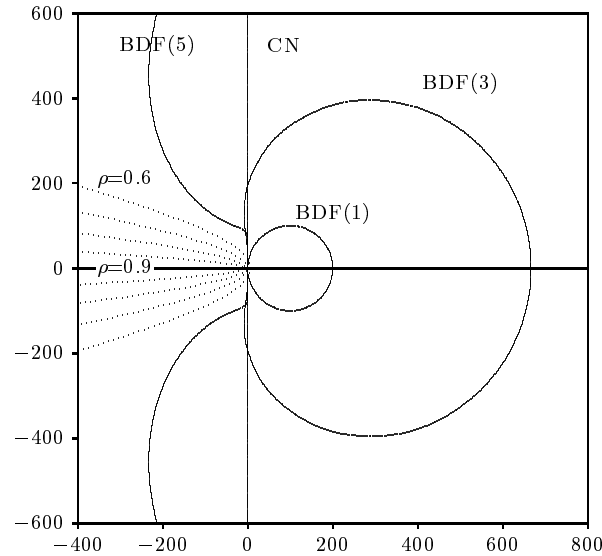


Figure 1: Spectral picture for (13) – optimal CSOR waveform relaxation – $h = 1/16$, $\tau = 1/100$.

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