Multigrid Waveform Relaxation
on Spatial Finite Element Meshes [*]

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Abstract. We analyze in this paper the convergence of multigrid waveform relaxation methods, applied to a system of ordinary differential equations, obtained by a finite element discretization of a parabolic initial boundary value problem. We restrict the discussion to linear systems and consider both the continuous-time and the discrete-time cases.

1   Introduction

Consider a parabolic partial differential equation (PDE)

$$\frac{\partial \mathbf{u}}{\partial t}(x,t) = \mathcal{L}\mathbf{u}(x,t) + \mathbf{f}(x,t) \quad x \in \Omega,\ t > 0 \ , \tag{1}$$

with a linear boundary condition and given initial values. $\mathcal{L}$ is a linear second order elliptic operator with time-independent coefficients and Ω is a compact spatial domain. A spatial finite element discretization on a mesh Ω_h with mesh size h transforms (1) into a linear system of ordinary differential equations (ODEs),

$$B\dot{u} + Au = f \ , \quad \text{with}\ \ u(0) = u_0 \ , \quad t > 0 \ , \tag{2}$$

where B is a non-singular constant-coefficient matrix.

The standard waveform relaxation method, also called the dynamic iteration method, is an iterative technique for solving systems of ordinary differential equations. Its background is in electrical network simulation, [4]. It differs from most standard iterative techniques in that it is a continuous-time method, iterating with functions. A discrete-time variant can be obtained by discretizing the

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continuous-time method in time, with, e.g., a linear multistep or Runge-Kutta method.

The discretization of (1) with finite differences results in a system of ODEs of the form (2) where B is the identity matrix. For such systems, the convergence of the standard waveform relaxation method has been studied exhaustively. Miekkala and Nevanlinna considered the continuous-time case in [7] and [9]. They formulated the convergence characteristics of the method in terms of a complex iteration matrix, obtained after Laplace-transforming the problem. Discrete-time results are obtained by the same authors in [8] and [10]. The multigrid acceleration of the waveform relaxation method was studied by Lubich and Ostermann in [5], where both the continuous-time and the discrete-time cases are considered. A complete survey and a discussion of a parallel implementation of these methods can be found in the work by Vandewalle, [11], and in the references cited therein. We also mention a paper by Miekkala, [6], where the convergence properties of the standard waveform relaxation method are studied for differential-algebraic systems of the form (2) where B is possibly singular.

In this paper, we shall concentrate on (2) with non-singular B and generalize some of the results of [5], [7] and [8]. This paper is a summary of [2] and [3] where further theoretical results and numerical experiments are reported. Miekkala's results are briefly recalled in section 2. These results are completed with a convergence analysis for the discrete-time standard waveform relaxation method. In section 3, the multigrid acceleration of this method for (2) is theoretically investigated. In section 4, we give some specific theoretical results for the heat equation, which are validated by numerical experiments.

2 Standard waveform relaxation

2.1 The continuous-time case

2.1.1 The continuous-time waveform relaxation operator

Consider the linear initial value problem (2) where B and A are complex $d \times d$ matrices, and u and f are $\mathbb{C}^d$-valued functions in time. B is assumed to be non-singular. Its solution is formally given by

$$u(t) = e^{-tB^{-1}A}u_0 + \int_0^t e^{(s-t)B^{-1}A}B^{-1}f(s)ds \ . \tag{3}$$

The waveform relaxation method for solving (2) is defined by introducing a splitting of the matrices B and A. With $B = M_B - N_B$ and $A = M_A - N_A$, the basic continuous-time waveform relaxation iteration can be written as

$$M_B \dot{u}^{(\nu)} + M_A u^{(\nu)} = N_B \dot{u}^{(\nu-1)} + N_A u^{(\nu-1)} + f \ , \ \ \text{with} \ \ u^{(\nu)}(0) = u_0 \ , \ \ t > 0 \ , \tag{4}$$

which is usually started by choosing the zeroth iterate $u^{(0)}(t) = u_0$, $t > 0$. We shall always assume M_B to be invertible. By using (3), we can rewrite (4) as an

explicit successive approximation scheme:

$$u^{(\nu)} = \mathcal{K}u^{(\nu-1)} + \varphi \ . \tag{5}$$

The right-hand side function φ can be found in the companion report, [2]. The *continuous-time waveform relaxation operator* $\mathcal{K}$ is found to be

$$\mathcal{K}u(t) = M_B^{-1}N_B u(t) + \mathcal{K}_c u(t) \ , \tag{6}$$

where $\mathcal{K}_c$ is a linear Volterra operator with a continuous kernel,

$$\mathcal{K}_c u(t) = \int_0^t k_c(t-s)u(s)ds \ , \tag{7}$$

$$k_c(t) = e^{-tM_B^{-1}M_A}M_B^{-1}(N_A - M_A M_B^{-1}N_B) \ . \tag{8}$$

Let $e^{(\nu)}$ be the error of the ν-th waveform relaxation iterate, i.e., $e^{(\nu)} = u^{(\nu)} - u$. It satisfies $e^{(\nu)} = \mathcal{K}e^{(\nu-1)}$. That is, it is the solution to

$$M_B \dot{e}^{(\nu)} + M_A e^{(\nu)} = N_B \dot{e}^{(\nu-1)} + N_A e^{(\nu-1)} \ , \quad \text{with} \ \ e^{(\nu)}(0) = 0 \ , \ \ t > 0 \ . \tag{9}$$

Denoting by $\tilde{e}^{(\nu)}(z)$ the Laplace-transform of $e^{(\nu)}$, we obtain by Laplace-transforming (9) that $\tilde{e}^{(\nu)} = \mathbf{K}(z)\tilde{e}^{(\nu-1)}$, with

$$\mathbf{K}(z) = (zM_B + M_A)^{-1}(zN_B + N_A) \ . \tag{10}$$

$\mathbf{K}(z)$ is called the dynamic iteration matrix of operator $\mathcal{K}$.

2.1.2 Convergence analysis

The convergence of the waveform relaxation operator is often studied in general Banach spaces. In particular, we consider the space of p-th power integrable Lebesgue measurable functions $L_p((0,\infty); \mathbb{C}^d)$, or $L_p(0,\infty)$ for short, with the usual mean p-norm, and the space of continuous functions $C([0,T]; \mathbb{C}^d)$, or $C[0,T]$, equipped with the maximum norm.

The spectral radius $\rho(\mathcal{U})$ of a bounded linear operator $\mathcal{U}$ in a complex normed linear space is characterized by

$$\rho(\mathcal{U}) = \lim_{n\to\infty} \sqrt[n]{||\mathcal{U}^n||} \ ,$$

and convergence of the general successive approximation scheme,

$$x^{(\nu)} = \mathcal{U}x^{(\nu-1)} + \varphi \ ,$$

is guaranteed if and only if $\rho(\mathcal{U}) < 1$.

In [2], we analyzed the convergence behaviour of the continuous-time waveform relaxation operator, both on finite and infinite time-intervals. The two main results are stated below. Supplied with some extra assumptions, Theorem 2 also holds for differential-algebraic systems with singular B, [6].

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Theorem 1 (finite time-interval) *Consider $\mathcal{K}$ as an operator in $C[0, T]$. Then,*

$$\rho(\mathcal{K}) = \rho(M_B^{-1} N_B) \ . \tag{11}$$

Theorem 2 (infinite time-interval) *Consider $\mathcal{K}$ as an operator in $L_p(0, \infty)$ with $1 \leq p \leq \infty$, and assume that all eigenvalues of $M_B^{-1} M_A$ have positive real parts. Then,*

$$\rho(\mathcal{K}) = \sup_{\xi \in I\!R} \rho(\mathbf{K}(i\xi)) \ . \tag{12}$$

Remark 1. The previous results can also be applied when B is the identity matrix, i.e., when we discretize our PDE using finite differences. Indeed, if $M_B = I$ and $N_B = 0$, we obtain the same results as in [7].

2.2 The discrete-time case

2.2.1 The discrete-time waveform relaxation operator

We recall the general linear multistep formula for calculating the solution to the ordinary differential equation $\dot{y} = f(t, y)$, $y(0) = y_0$,

$$\frac{1}{\tau} \sum_{j=0}^{k} \alpha_j y_{n+j} = \sum_{j=0}^{k} \beta_j f_{n+j} \ .$$

τ denotes a constant step-size; α_j and β_j are real constants, and y_j approximates the ODE solution at time-level $t = j\tau$. The fully discrete solution $\{y_i\}_{i=0}^{N_t}$, with N_t the number of time-steps, will be denoted further by y_τ. The characteristic polynomials of the linear multistep method are $a(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j$ and $b(\xi) = \sum_{j=0}^{k} \beta_j \xi^j$. We shall adhere to the usual assumptions: $a(\xi)$, $b(\xi)$ have no common roots; $a(1) = 0$, $a'(1) = b(1)$; all roots of $a(\xi)$ are inside the closed unit disk, and every root with modulus one is simple.

Application of the linear multistep formula to (9) leads to

$$\frac{1}{\tau} \sum_{j=0}^{k} \alpha_j M_B e_{n+j}^{(\nu)} + \sum_{j=0}^{k} \beta_j M_A e_{n+j}^{(\nu)} = \frac{1}{\tau} \sum_{j=0}^{k} \alpha_j N_B e_{n+j}^{(\nu-1)} + \sum_{j=0}^{k} \beta_j N_A e_{n+j}^{(\nu-1)} \ , \tag{13}$$

with $n \geq 0$ and $e_\tau^{(\nu)} = u_\tau^{(\nu)} - u_\tau$ the error of the ν-th discrete waveform relaxation iterate. For simplicity's sake, we assume that there are k fixed starting values supplied, i.e., $u_j^{(\nu)} = u_j^{(\nu-1)} = u_j$ for $j < k$. So, we do not iterate on the starting values. Iteration (13) can be rewritten as $e_\tau^{(\nu)} = \mathcal{K}_\tau e_\tau^{(\nu-1)}$. In [3] it is shown that the *discrete-time waveform relaxation operator* $\mathcal{K}_\tau$ is a discrete convolution operator:

$$(\mathcal{K}_\tau x_\tau)_j = (k_\tau \star x_\tau)_j = \sum_{i=0}^{j} k_{j-i} x_i \ . \tag{14}$$

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Its kernel, k_τ, is related to the dynamic iteration matrix of $\mathcal{K}_\tau$ by the ξ-transform (or discrete Laplace-transform),

$$\mathbf{K}_\tau(\xi) := \sum_{i=0}^{\infty} k_i \xi^{-i} = \left(\frac{1}{\tau} \frac{a}{b}(\xi) M_B + M_A \right)^{-1} \left(\frac{1}{\tau} \frac{a}{b}(\xi) N_B + N_A \right) , \qquad (15)$$

which is equal to $\mathbf{K}(z)$ for $z = \frac{1}{\tau} \frac{a}{b}(\xi)$.

2.2.2 Convergence results

The convergence of the discrete waveform relaxation is studied in the Banach spaces of p-summable sequences $l_p(0..N_t)$, equipped with the mean p-norm.

The discrete versions of Theorem 1 and 2 are given below. Their proof is given in [3].

Theorem 3 (finite time-interval) *Consider $\mathcal{K}_\tau$ as an operator in $l_p(0..N_t)$ with $1 \leq p \leq \infty$. Let the number of time-steps, N_t, be finite and suppose that $\frac{\alpha_k}{\beta_k} \notin \sigma(-\tau M_B^{-1} M_A)$. Then,*

$$\rho(\mathcal{K}_\tau) = \rho\left(\mathbf{K}(\frac{1}{\tau} \frac{\alpha_k}{\beta_k}) \right) . \qquad (16)$$

Theorem 4 (infinite time-interval) *Consider $\mathcal{K}_\tau$ as an operator in $l_p(\mathbb{N})$ with $1 \leq p \leq \infty$ and suppose $\sigma(-\tau M_B^{-1} M_A) \subset int\ S$. Then,*

$$\rho(\mathcal{K}_\tau) = \sup_{|\xi|=1} \rho\left(\mathbf{K}\left(\frac{1}{\tau} \frac{a}{b}(\xi) \right) \right) . \qquad (17)$$

3 Multigrid waveform relaxation

3.1 The continuous-time case

3.1.1 The continuous-time two-grid waveform relaxation operator

Multigrid methods are known to be very efficient solvers for elliptic partial differential equations. We refer to [1] and [12] for a detailed analysis. The principle can easily be extended to time-dependent problems by choosing all the operations in the multigrid cycle as operations on functions. A two-grid cycle for the initial value problem (2) is stated below. It is defined on two nested grids Ω_H and Ω_h, with $\Omega_H \subset \Omega_h$, and determines a new iterate $u^{(\nu)}$ from the former waveform $u^{(\nu-1)}$ in three steps: pre-smoothing, coarse grid correction, and post-smoothing. In the following, the subscripts h and H are used to denote fine and coarse grid quantities respectively.

- **Pre-smoothing.** Set $x^{(0)} = u^{(\nu-1)}$, and perform ν_1 waveform relaxation steps: for $\nu = 1, 2, \ldots, \nu_1$, solve

$$M_{B_h} \dot{x}^{(\nu)} + M_{A_h} x^{(\nu)} = N_{B_h} \dot{x}^{(\nu-1)} + N_{A_h} x^{(\nu-1)} + f_h , \qquad (18)$$

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with $x^{(\nu)}(0) = u_0$, $t > 0$.

- **Coarse grid correction.** Compute the defect,

$$
\begin{aligned}
d_h &= B_h \dot{x}^{(\nu_1)} + A_h x^{(\nu_1)} - f_h \\
&= N_{B_h}(\dot{x}^{(\nu_1-1)} - \dot{x}^{(\nu_1)}) + N_{A_h}(x^{(\nu_1-1)} - x^{(\nu_1)}) \; .
\end{aligned}
$$

Solve the coarse grid equivalent of the defect equation,

$$
B_H \dot{v}_H + A_H v_H = r d_h \; , \quad \text{with} \;\; v_H(0) = 0 \; , \quad t > 0 \; , \tag{19}
$$

with $r : \Omega_h \to \Omega_H$ the restriction operator. Then interpolate the correction v_H to Ω_h, and correct the current approximation,

$$
\bar{x} = x^{(\nu_1)} - p v_H \; ,
$$

with $p : \Omega_H \to \Omega_h$ the prolongation operator.

- **Post-smoothing.** Perform ν_2 iterations of type (18), starting with $x^{(0)} = \bar{x}$, and set $u^{(\nu)} = x^{(\nu_2)}$.

Since (19) is formally equal to (2), this two-grid cycle can be applied in a recursive way to obtain a multigrid cycle.

The two-grid cycle can be written as a successive approximation scheme:

$$
u^{(\nu)} = \mathcal{M} u^{(\nu-1)} + \varphi \; . \tag{20}
$$

The *continuous-time two-grid waveform relaxation operator* $\mathcal{M}$ is given by

$$
\mathcal{M}u(t) = (M_{B_h}^{-1} N_{B_h})^{\nu_2} (I - p B_H^{-1} r B_h)(M_{B_h}^{-1} N_{B_h})^{\nu_1} u(t) + \mathcal{M}_c u(t) \; . \tag{21}
$$

$\mathcal{M}_c$ is a linear Volterra convolution operator with kernel m_c, whose Laplace-transform is denoted by $\mathbf{M_c}(z)$. Let $e^{(\nu)} = u^{(\nu)} - u$ be the error of the ν-th two-grid waveform relaxation iterate. This error satisfies the relation $e^{(\nu)} = \mathcal{M} e^{(\nu-1)}$. Laplace-transforming yields

$$
\tilde{e}^{(\nu)} = \mathbf{M}(z) \tilde{e}^{(\nu-1)} \; , \tag{22}
$$

with $\mathbf{M}(z)$ the two-grid dynamic iteration matrix of $\mathcal{M}$,

$$
\begin{aligned}
\mathbf{M}(z) &= \mathbf{S}^{(\nu_2)}(z)(I - p(z B_H + A_H)^{-1} r(z B_h + A_h))\mathbf{S}^{(\nu_1)}(z) \; , \tag{23} \\
\mathbf{S}(z) &= (z M_{B_h} + M_{A_h})^{-1}(z N_{B_h} + N_{A_h}) \; . \tag{24}
\end{aligned}
$$

3.1.2   Convergence analysis

The following theorems are the two-grid analogues of Theorem 1 and Theorem 2, and are proven in [2].

Theorem 5 (finite time-interval) *Consider $\mathcal{M}$ as an operator in $C[0,T]$. Then,*

$$\rho(\mathcal{M}) = \rho\left((M_{B_h}^{-1}N_{B_h})^{\nu_2}(I - pB_H^{-1}rB_h)(M_{B_h}^{-1}N_{B_h})^{\nu_1}\right) \ . \tag{25}$$

Theorem 6 (infinite time-interval) *Consider $\mathcal{M}$ as an operator in $L_p(0,\infty)$ with $1 \leq p \leq \infty$, and assume that all eigenvalues of $B_H^{-1}A_H$ and $M_{B_h}^{-1}M_{A_h}$ have positive real parts. Then,*

$$\rho(\mathcal{M}) = \sup_{\xi \in I\!R} \rho(\mathbf{M}(i\xi)) \ . \tag{26}$$

Remark 2. The previous results can also be applied when both B_H and B_h are identity matrices, i.e., when we discretize our PDE using finite differences. In that case, we obtain the same results as in [11, Th. 3.4.1] and [5, Prop. 1].

3.2 The discrete-time case

3.2.1 The discrete-time two-grid waveform relaxation operator

We discretize the two-grid operator in time using a linear multistep formula, and we assume again that we do not iterate on the k given starting values. The error $e_\tau^{(\nu)} = u_\tau^{(\nu)} - u_\tau$ of the fully discrete ν-th two-grid iterate then satisfies $e_\tau^{(\nu)} = \mathcal{M}_\tau e_\tau^{(\nu-1)}$. In an analogous manner as in the previous section, it can be shown that the *discrete-time two-grid waveform relaxation operator* $\mathcal{M}_\tau$ is a discrete convolution operator. The ξ-transform of its kernel m_τ equals

$$\mathbf{M}_\tau(\xi) := \sum_{i=0}^{\infty} m_i \xi^{-i} = \mathbf{M}\left(\frac{1}{\tau}\frac{a}{b}(\xi)\right) \ . \tag{27}$$

3.2.2 Convergence analysis

The convergence of the discrete-time two-grid waveform relaxation is studied in [3]. The following results are very similar to the results obtained for the discrete-time standard waveform relaxation operator.

Theorem 7 (finite time-interval) *Consider $\mathcal{M}_\tau$ as an operator in $l_p(0..N_t)$ with $1 \leq p \leq \infty$. Let the number of time-steps, N_t, be finite and assume none of the poles of $\mathbf{M}(z)$ is equal to $\frac{\alpha_k}{\beta_k}$. Then,*

$$\rho(\mathcal{M}_\tau) = \rho\left(\mathbf{M}(\frac{1}{\tau}\frac{\alpha_k}{\beta_k})\right) \ . \tag{28}$$

Theorem 8 (infinite time-interval) *Consider $\mathcal{M}_\tau$ as an operator in $l_p(I\!N)$ with $1 \leq p \leq \infty$ and suppose all the poles of $\mathbf{M}(z)$ are in the interior of the scaled stability region $\frac{1}{\tau}S$. Then,*

$$\rho(\mathcal{M}_\tau) = \sup_{|\xi|=1} \rho\left(\mathbf{M}(\frac{1}{\tau}\frac{a}{b}(\xi))\right) \ . \tag{29}$$

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4 Model problem analysis and numerical results

In this section, we verify our theoretical results on the basis of numerical experiments with two model problems.

Model problem 1: the one-dimensional heat equation

$$\frac{\partial \mathbf{u}}{\partial t}(x, t) = \frac{\partial^2 \mathbf{u}}{\partial x^2}(x, t) \quad x \in [0, 1] \; , \quad t > 0 \; , \tag{30}$$

completed with homogeneous Dirichlet boundary conditions and an initial condition. The analytical solution is given by $\mathbf{u}(x, t) = \sin(\pi x) \exp(-\pi^2 t)$.

Model problem 2: the two-dimensional heat equation

$$\frac{\partial \mathbf{u}}{\partial t}(x, y, t) = \frac{\partial^2 \mathbf{u}}{\partial x^2}(x, y, t) + \frac{\partial^2 \mathbf{u}}{\partial y^2}(x, y, t) \quad (x, y) \in [0, 1] \times [0, 1] \; , \quad t > 0 \; , \tag{31}$$

completed with Dirichlet boundary conditions and an initial condition such that the analytical solution is given by $\mathbf{u}(x, y, t) = 1 + \sin(\pi x/2) \sin(\pi y/2) \exp(-\pi^2 t/2)$.

Discretizing these model problems on a discrete mesh Ω_h yields systems of the form (2). The matrices B and A are listed for several finite element basis functions in [2].

4.1 Gauss-Seidel waveform relaxation

4.1.1 Theoretical results

In order to determine the spectral radius of the continuous-time Gauss-Seidel waveform relaxation method, the spectral radius of $\mathbf{K}(z)$ is to be calculated for every value of z along the imaginary axis. This is generally a very difficult task. However, there are some cases where $\rho(\mathcal{K})$ can be calculated explicitly.

Assume B and A to be decomposed as $B = -L_B + D_B - U_B$ and $A = -L_A + D_A - U_A$, with D_B and D_A diagonal matrices, L_B and L_A strictly lower and U_B and U_A strictly upper triangular matrices. Let $\mathcal{K}_{GS}$ and $\mathcal{K}_{JAC}$ denote the Gauss-Seidel and Jacobi waveform relaxation operator respectively. Following [9], their dynamic iteration matrices are given by:

$$\begin{aligned}
\mathbf{K_{GS}}(z) &= (z(D_B - L_B) + (D_A - L_A))^{-1}(zU_B + U_A) \; , \\
\mathbf{K_{JAC}}(z) &= (zD_B + D_A)^{-1}(z(L_B + U_B) + (L_A + U_A)) \; .
\end{aligned}$$

Lemma 1 *Assume that all eigenvalues of $(D_B - L_B)^{-1}(D_A - L_A)$ have positive real parts. Let A and B be such that $(zB + A)$ is a consistently ordered matrix for $Re(z) \geq 0$. Then, in $L_p(0, \infty)$ with $1 \leq p \leq \infty$,*

$$\rho(\mathcal{K}_{GS}) = \rho(\mathcal{K}_{JAC})^2 \; . \tag{32}$$

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This Lemma is proven in [2]. Discretizing model problem 1 with linear basis functions yields

$$\rho(\mathbf{K_{JAC}}(z)) = \left| \frac{-2zh^2 + 12}{4zh^2 + 12} \right| \cos(\pi h) .$$

Hence, as a consequence of Theorem 2, equation (12), we find that

$$\begin{aligned} \rho(\mathcal{K}_{JAC}) &= \sup_{\xi \in \mathbb{R}} \left| \frac{-2i\xi h^2 + 12}{4i\xi h^2 + 12} \right| \cos(\pi h) = \cos(\pi h) \\ &\approx 1 - \pi^2 h^2/2 . \end{aligned}$$

Since the assumptions of Lemma 1 are satisfied, we have for small h

$$\rho(\mathcal{K}_{GS}) \approx 1 - \pi^2 h^2 . \tag{33}$$

4.1.2   Numerical results

For the time-discretization of (2), we use the Crank-Nicolson formula. When the time-step τ is taken sufficiently small, the continuous-time theoretical results should fit the obtained results of the numerical experiments. For the ν-th waveform relaxation iterate, we determine the l_2-norm of the defect $d_\tau^{(\nu)} = B\dot{u}_\tau^{(\nu)} + Au_\tau^{(\nu)} - f_\tau$. The iteration convergence factor is then calculated as

$$\rho^{(\nu)} = \frac{||d_\tau^{(\nu)}||_2}{||d_\tau^{(\nu-1)}||_2} .$$

After a sufficiently large number of iterations this factor takes a nearly constant value, the *averaged convergence factor*.

The numerical results for model problem 1, discretized using linear basis functions, are given in Table 1. Even though the time-interval in this experiment is finite, the measured convergence factors closely match the infinite interval theoretical spectral radii of (33). For a discussion of this phenomenon, we refer to [11].

Table 1 also reports the averaged convergence factors for the same problem, discretized using quadratic basis functions. Observe that these factors seem to satisfy a relation of the form

$$\rho(\mathcal{K}_{GS}) \approx 1 - O(h^2) ,$$

although no explicit theoretical formula was found. The same is true for the results of Table 2, where we reported the averaged convergence factors for model problem 2, discretized with bilinear basis functions on an equidistant rectangular grid.

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h	$1/5$	$1/10$	$1/15$	$1/20$	$1/30$	$1/40$
Linear	0.630	0.898	0.953	0.973	0.988	0.993
$1 - \pi^2 h^2$	0.605	0.901	0.956	0.975	0.989	0.994
Quadratic	0.917	0.978	0.991	0.995	0.998	0.999

Table 1: Gauss-Seidel waveform relaxation averaged convergence factors for the one-dimensional heat equation.

h	$1/5$	$1/10$	$1/15$	$1/20$	$1/30$	$1/40$
Bilinear	0.530	0.854	0.933	0.962	0.983	0.990

Table 2: Gauss-Seidel waveform relaxation averaged convergence factors for the two-dimensional heat equation.

4.2   Multigrid waveform relaxation

4.2.1   Theoretical results

The coarse grid Ω_H is derived from the fine grid Ω_h by standard coarsening ($H = 2h$). For the prolongation operator $p : \Omega_H \to \Omega_h$, we use piecewise linear interpolation. The restriction operator $r : \Omega_h \to \Omega_H$ is then defined as the transpose of the prolongation operator, $r = p^t$, see e.g. [1, p. 66] or [12, p. 70].

Lemma 2 *The two-grid operator $\mathcal{M}$ for the one-dimensional heat equation, discretized using linear basis functions, with red-black Gauss-Seidel smoothing, and with the prolongation and restriction operator defined as above, satisfies*

$$\rho(\mathcal{M}) \leq \sqrt{3}\sqrt{\eta_0(2\nu - 1)}\ , \quad with \ \ \eta_0(\nu) = \frac{\nu^\nu}{(\nu + 1)^{\nu+1}}\ , \tag{34}$$

for $\nu = \nu_1 + \nu_2 \geq 1$.

The lemma states that the spectral radius of the two-grid operator $\mathcal{M}$ is bounded by a constant, independent of h. A table of the bound is given in Table 3. The proof is given in [2].

4.2.2   Numerical results

Since the bound in Lemma 2 is not optimal, we numerically computed the spectral radius of the two-grid operator by evaluation of (26), for $\nu = 2$ and for several values of h. These results are reported in the first line of Table 4, and are compared to the averaged convergence factors of different multigrid cycles. We use $V(\nu_1, \nu_2)$

ν	1	2	3	4
$\sqrt{3}\sqrt{\eta_0(2\nu-1)}$	0.866	0.563	0.448	0.384

Table 3: Values of the upper bound for $\rho(\mathcal{M})$.

and $W(\nu_1, \nu_2)$ as shorthands for a standard V-cycle and W-cycle with ν_1 pre-smoothing, ν_2 post-smoothing steps, and standard coarsening down to a grid with mesh size $h = 0.5$. The other parameters are as above: red-black Gauss-Seidel smoothing, linear interpolation and corresponding restriction.

Finally, we report some averaged convergence factors for the two-dimensional model problem, discretized using bilinear basis functions on an equidistant rectangular grid.

h	1/8	1/16	1/32	1/64
$\rho_{\mathrm{num}}(\mathcal{M})$	0.217	0.263	0.276	0.280
$V(1,1)$	0.228	0.301	0.325	0.332
$W(1,1)$	0.211	0.253	0.265	0.268

Table 4: Multigrid waveform relaxation averaged convergence factors for the one-dimensional heat equation.

h	1/4	1/8	1/16	1/32
$V(1,1)$	0.138	0.299	0.352	0.361
$W(1,1)$	0.138	0.294	0.343	0.355

Table 5: Multigrid waveform relaxation averaged convergence factors for the two-dimensional heat equation.

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